# Properties of FFT Windows Used in Stable32

Introduction

The Stable32 Power function offers the two classic Hanning and Hamming FFT windows for its periodogram methods of calculating power spectral density. Corrections are made to the resulting PSD values based on the properties of these windowing functions. The Stable32 Power function is intended primarily for the analysis of noise. If the user wishes to establish the amplitude of a discrete spectral component, it is necessary to multiply the PSD value by the effective noise bandwidth used in the analysis. This paper describes the properties of the Stable32 FFT windows and the way to convert a PSD value to the amplitude of a discrete component.

FFT Windows

The purpose of an FFT window function is to reduce spectral leakage from adjacent Fourier frequency bins and to thereby improve the dynamic range of the analysis. For the analysis of incoherent noise, amplitude flatness is not a particular concern, and a non-flattopped window is both acceptable and has a faster rolloff rate. Spectral leakage is generally not a big problem either, except possibly near the low frequency end, or for highly-divergent noise types (e.g., random walk FM noise phase data, where an analysis of the frequency data is recommended). Stable32 offers the use of 0 through 3 windowings of these window functions. The zero option is the same as None (i.e., a rectangular window). In general, windowing has little effect on the analysis of white noise.

The Hanning and Hamming window functions are shown in the equations and plots below (shown arbitrarily over a full range of 1000 data points). The plots show the 1, 2 and 3 windowings offered by the Stable32 Power function. Multiple windowings have a narrower response and lower sidelobes, but use the data less efficiently (the multi-taper method is recommended as a more efficient alternative).
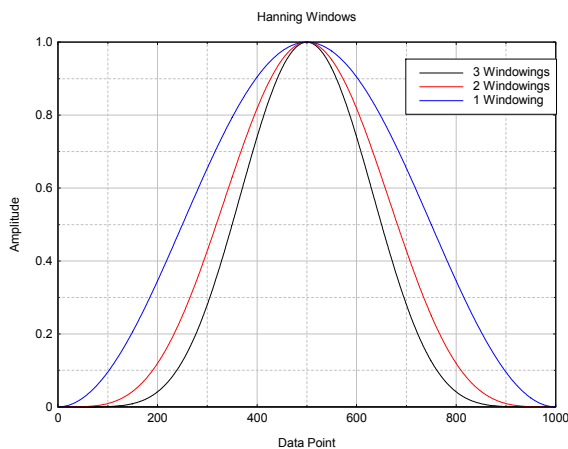
$$w(n) = 0.5\{1 - \cos(2\pi \cdot n / N - 1)\}, \; n = 0 \text{ to } N - 1$$

Hanning Window
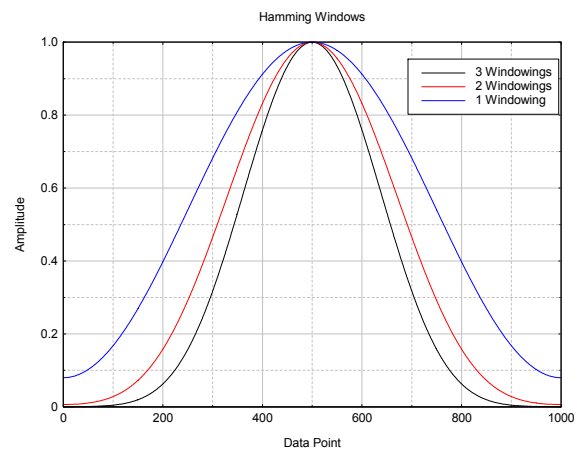
$$w(n) = 0.54 - 0.46\cos(2\pi \cdot n / N - 1), \; n = 0 \text{ to } N - 1$$

Hamming Window

Amplitude Response

The numerical amplitude response of these windows is as follows:

Numerical Amplitude Response of Stable32 Window Functions				
Window Type	# Windowings			
	0	1	2	3
Hanning	1.0000	0.5000	0.3750	0.3125
Hamming	1.0000	0.5400	0.3974	0.3288

These values are the numerical gain amplitude (not power gain) as determined by averaging the amplitude response over the whole window. For example, the values for 1 windowing are the same as those in the Coherent Gain column of Table I of the Harris paper [1]. For use as correction factors for a PSD plot, they could be squared, divided by the appropriate equivalent noise bandwidth, ENBW in bins, and the negative of their common log taken. But the more direct approach is to use the incoherent power gain as described below.

Sidelobe Levels

The sidelobe attenuation of these windows is shown in the following table (as determined by the approximation method described in Reference [3]).

Sidelobe Response of Stable32 Window Functions, dB				
Window Type	# Windowings			
	0 (Rectangular)	1	2	3
Hanning	-13	-32	-47	-61
Hamming	-13	-43	-49	-67
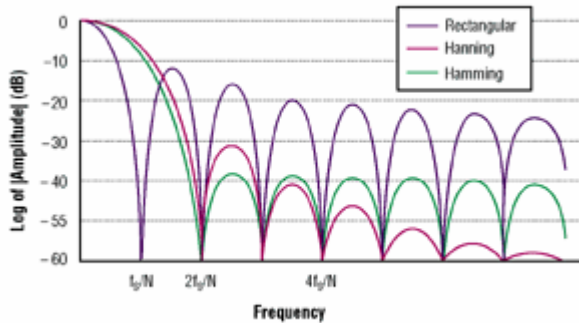
The sidelobe values for 1 windowing are the same as those in the Highest Sidelobe Level column of the Harris paper [1] and are plotted at the left [from 3]. More windowings improve the sidelobe performance, but not as much as other windowing functions specifically developed for that purpose [2], [4].

Incoherent and Coherent Power Gains

The incoherent and coherent power gains of these windows are the appropriate respective correction factors for a power spectral density plot showing noise and a power spectral plot showing discrete signals. These are respectively equal to the normalized sum of the squares of the windowing values, and the square of the normalized sum of the windowing values, as shown in the following expressions.

$$\text{Incoherent Power Gain} = \frac{1}{N}\sum_{i=1}^{N} w(i)^2 \qquad\qquad \text{Coherent Power Gain} = \left[\frac{1}{N}\sum_{i=1}^{N} w(i)\right]^2$$

The coherent power gain is also equal to the square of the amplitude responses shown above. Of particular interest is the effective noise bandwidth of the windows, which is equal to the ratio of these power gains:

ENBW = Incoherent Power Gain / Coherent Power Gain

These quantities are shown in the table below for the 1-3 Hanning and Hamming windows used by the Stable32 program.

Numerical Power Gains and Noise Bandwidth of Stable32 Window Functions,							
Window Type	Power Gain Type	# Windowings					
		1		2		3	
		Power Gain	ENBW bins	Power Gain	ENBW bins	Power Gain	ENBW bins
Hanning	Incoherent	0.3750	1.500	0.2734	1.944	0.2256	2.310
	Coherent	0.2500		0.1406		0.09765	
Hamming	Incoherent	0.3974	1.363	0.2869	1.817	0.2361	2.184
	Coherent	0.2916		0.1579		0.1081	

It is seen that using a larger number of windowings increases the noise bandwidth while improving the overall dynamic range because of smaller sidelobes.

PSD values obtained from time series data by an FFT process must be corrected for the incoherent power gain (actually a loss) of the window. Those corrections are most conveniently applied in log form as shown in the table below:

Log PSD Correction Factor = - log (Incoherent Power Gain)

Log PSD Correction Factors for Stable32 Window Functions			
Window Type	# Windowings		
	1	2	3
Hanning	0.426	0.563	0.647
Hamming	0.401	0.542	0.627

These log corrections are used directly for the Stable32 $S_x(f)$, $S_\phi(f)$, and $S_y(f)$ plots, and multiplied by 10 for the $\mathcal{L}(f)$ plot.

Discrete Components

The Stable32 Power function supports four types of PSD plot, three for phase data, and one for frequency data, which are intended primarily for the analysis of noise. If the spectra show a discrete component, its amplitude on a corresponding PS plot can be determined by using the appropriate ENBW (in bins) shown above along with the PSD frequency resolution, f_{res}, (in Hz/bin) of a Fourier frequency bin (called Fourier Interval in the Power function dialog box).

Amplitude of discrete component in PS plot = PSD value · f_{res} · ENBW

References

1. F.J Harris, "On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform", *Proc. IEEE*, January 1978, pp. 51-83.
2. G. Heinzel, A. Rudiger, and R. Schilling, "Spectrum and Spectral Density Estimation by the Discrete Fourier transform (DFT), Including a Comprehensive List of Window Functions and Some New Flat-Top Windows", Max Plank Institute, February 2002.
3. R. Lyons, "Windowing Functions Improve FFT Results", *Test and Measurement World*, Parts I and II, June and September 1998.
4. A.H. Nuttall, " Some Windows with Very Good Sidelobe Behavior", IEEE Transactions on Acoustics, Speech, and Signal Processing, Volume 29, Issue 1, February 1981, pp. 84 – 91.